UNIT – II

## UNIT – II

### Planned Topics

<table>
<thead>
<tr>
<th>Lectures</th>
<th>Faculty Name: N V Nagendram</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 Introduction</td>
<td></td>
</tr>
<tr>
<td>2.2 Quadratic form</td>
<td></td>
</tr>
<tr>
<td>2.3 Theorem Statement</td>
<td></td>
</tr>
<tr>
<td>2.4 Matrix quadratic Form - Rules to write the matrix of a Quadratic form</td>
<td></td>
</tr>
<tr>
<td>2.5 Linear Transformation of a Quadratic form</td>
<td></td>
</tr>
<tr>
<td>2.6 Orthogonal Transformation</td>
<td></td>
</tr>
<tr>
<td>2.7 Rank of a quadratic Form – Canonical form or Normal form of a Quadratic Form</td>
<td></td>
</tr>
<tr>
<td>2.8 Index of a Quadratic Form</td>
<td></td>
</tr>
<tr>
<td>2.9 Theorem</td>
<td></td>
</tr>
<tr>
<td>2.10 Signature of a Quadratic Form</td>
<td></td>
</tr>
<tr>
<td>2.11 Nature of Quadratic Form</td>
<td></td>
</tr>
<tr>
<td>2.12 Sylvester’s law of Inertia</td>
<td></td>
</tr>
<tr>
<td>2.13 Reduction of Quadratic Form to Canonical Form</td>
<td></td>
</tr>
<tr>
<td>2.14 Reduction of Quadratic Form to Normal Form by Orthogonal Transformation</td>
<td></td>
</tr>
<tr>
<td>2.15 Reduction of Quadratic Form to Canonical Form by Lagrange’s Method</td>
<td></td>
</tr>
<tr>
<td>2.16 Solution of Algebraic and Transcendental Equations</td>
<td></td>
</tr>
<tr>
<td>2.17 Method of False Position</td>
<td></td>
</tr>
<tr>
<td>2.18 Newton – Raphson Method</td>
<td></td>
</tr>
</tbody>
</table>
Definition: Quadratic Form: An expression of the form \( \sum_{i,j=1}^{n} a_{ij} x_i x_j \), where \( a_{ij} \) are elements of a field \( F \) is called a “Quadratic Form” in the \( n \) – variables \( x_1, x_2, \ldots, x_n \) over a field \( F \).

Definition: Real Quadratic Form: An expression of the form \( \sum_{i,j=1}^{n} a_{ij} x_i x_j \), where \( a_{ij} \) are all reals is called a “Real Quadratic Form” in the \( n \) – variables \( x_1, x_2, \ldots, x_n \).

Example 1: \( ax^2 + 2hxy + by^2 \) is Real Quadratic Form in the two variables \( x \) and \( y \).
Example 2: \( 2x^2 + 7xy + 5y^2 \) is Real Quadratic Form in the two variables \( x \) and \( y \).
Example 3: \( 2x^2 - y^2 + 2z^2 + 2yz - 4zx + 6xy \) is Real Quadratic Form in the three variables \( x, y, \) and \( z \).
Example 4: \( x_1^2 - 2x_2^2 + 4x_3^2 - 4x_4^2 - 2x_1x_2 + 3x_1x_4 + 4x_2x_3 - 5x_3x_4 \) is a Real Quadratic Form in the four variables \( x_1, x_2, x_3, x_4 \).

Note: Every Quadratic Form over a field \( F \) in \( n \) – variables \( x_1, x_2, x_3, \ldots, x_n \) can be expressed as in the form \( X' B X \) where \( X = [x_1, x_2, x_3, \ldots, x_n]^T \) is a column matrix vector and \( B \) is a symmetric matrix of order \( n \) over a field \( F \).

i.e., \( X' B X \approx \sum_{i,j=1}^{n} b_{ij} x_i x_j \approx \sum_{i,j=1}^{n} a_{ij} x_i x_j \)

Definition: Matrix of Quadratic Form: If \( \phi = \sum_{i,j=1}^{n} a_{ij} x_i x_j \) is a Quadratic Form in \( n \) – variables \( x_1, x_2, x_3, \ldots, x_n \). Then there exists a unique symmetric matrix \( B \) of order \( n \) such that \( \phi = X' BX \) where \( X = [x_1, x_2, x_3, \ldots, x_n]^T \). The symmetric matrix \( B \) is called “The matrix of the Quadratic Form” \( \sum_{i,j=1}^{n} a_{ij} x_i x_j \).

Definition: Quadratic Form corresponding to a symmetric Matrix: Let \( A = [a_{ij}]_{n \times n} \) be a symmetric matrix over the field \( F \) and let \( X = [x_1, x_2, x_3, \ldots, x_n]^T \) be a column matrix vector.

Then \( X' A X \) determines a unique quadratic form \( \sum_{i,j=1}^{n} a_{ij} x_i x_j \) in \( n \) – variables say \( x_1, x_2, x_3, \ldots, x_n \).
Problem #1 Write down the matrix of the Quadratic Form and verify that they can be written as Matrix products $X^TAX$ (i) $x_1^2 - 2x_1 x_2 + x_2^2$ (ii) $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$.

Problem #2 Write down the matrix of the Quadratic Form and verify that they can be written as Matrix products $X^TAX$. $4x_1^2 + 2x_2^2 - 3x_3^2 + 2x_1 x_2 + 4x_1 x_3$.

Problem #3 Write down the matrix of the Quadratic Form and verify that they can be written as Matrix products $X^TAX$. $2xy + 2yz + 2zx$.

Problem #4 Write down the matrix of the Quadratic Form and verify that they can be written as Matrix products $X^TAX$. $x_1^2 - 2x_2^2 + 4x_3^2 - 4x_4^2 - 2x_1 x_2 + 3x_1 x_4 + 4x_2 x_3 - 5x_3 x_4$.

Problem #5 Write down the matrix of the Quadratic Form and verify that they can be written as Matrix products $X^TAX$. $x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6zx$.

Problem #6 write down the Quadratic Form corresponding to Matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}$

[Ans. $x_1^2 + x_3^2 + 4x_1 x_2 + 6x_1 x_3 + 6x_2 x_3$]

Problem #6 write down the Quadratic Form corresponding to Matrix $A = \begin{bmatrix} 1 & -9 \\ -9 & 5 \end{bmatrix}$

[Ans. $x_1^2 + 5x_2^2 - 18x_1 x_2$]

Problem #6 write down the Quadratic Form corresponding to Matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & -2 & 1 \\ 5 & 1 & 6 \end{bmatrix}$

[Ans. $x_1^2 - 2x_2^2 + 6x_3^2 + 6x_1 x_2 + 10x_1 x_3 + 2x_2 x_3$]
Problem #1 Write down the matrix of the Quadratic Form and verify that they can be written as Matrix products $X^TAX$ (i) $x_1^2 - 2x_1x_2 + x_2^2$  (ii) $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$.

Solution: Let the given Quadratic Form (i) $x_1^2 - 2x_1x_2 + x_2^2$ can be written as $x_1x_1 - x_1x_2 - x_2x_1 + x_2x_2$.

Let $A$ be matrix form of given Quadratic Form (Q.F.) Then, $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then $X' = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$

We have $X'A = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 - x_1 + x_2 \end{bmatrix}$

$\therefore$ $X'AX = \begin{bmatrix} x_1 - x_2 - x_1 + x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1 - x_2)x_1 + (-x_1 + x_2)x_2$

$\therefore$ $X'AX = x_1^2 - 2x_1x_2 + x_2^2$

Let the given Quadratic Form (ii) $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ can be written as $ax + by + cz + fyz + gzx + gxz + hxy + hxy$.

Let $A$ be matrix form of given Quadratic Form (Q.F.) Then, $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then $X' = \begin{bmatrix} x & y & z \end{bmatrix}$

We have $X'A = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = \begin{bmatrix} ax + hy + gz & xh + by + fz & xg + fy + zc \end{bmatrix}$

$\therefore$ $X'AX = \begin{bmatrix} ax + hy + gz & xh + by + fz & xg + fy + zc \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (ax + hy + gz)x + (xh + by + fz)y + (xg + fy + zc)z$

$\therefore$ $X'AX = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$

Hence the solution.
Problem #2 Write down the matrix of the Quadratic Form and verify that they can be written as Matrix products $X^T A X$. $4x_1^2 + 2x_2^2 - 3x_3^2 + 2x_1x_2 + 4x_1x_3$.

Solution: Let the given Quadratic Form $4x_1^2 + 2x_2^2 - 3x_3^2 + 2x_1x_2 + 4x_1x_3$ can be written as $4x_1x_1 + 2x_2x_2 - 3x_3x_3 + + x_1x_2 + x_2x_1 + 2x_1x_3 + 2x_3x_1 + 0x_3x_2 + 0x_2x_3$

Let $A$ be matrix form of given Quadratic Form (Q.F.) Then, $A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & -3 \end{bmatrix}$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $X' = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$

$\Rightarrow X'A = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 4x_1 + x_2 + 2x_3 & x_1 + 2x_2 + 0x_3 & 2x_1 + 0x_2 - x_3 \end{bmatrix}$

$\therefore X'AX = \begin{bmatrix} 4x_1 + x_2 + 2x_3 & x_1 + 2x_2 + 0x_3 & 2x_1 + 0x_2 - x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= (4x_1 + x_2 + 2x_3)x_1 + (x_1 + 2x_2 + 0x_3)x_2 + (2x_1 + 0x_2 - x_3)x_3$

$\therefore X'AX = 4x_1^2 + 2x_2^2 - 3x_3^2 + 2x_1x_2 + 4x_1x_3$. Hence the solution.

Problem #3 Write down the matrix of the Quadratic Form and verify that they can be written as Matrix products $X^T A X$. $2xy + 2yz + 2zx$.

Solution: Let the given Quadratic Form $321 x + 4 xx$ can be written

as $4x_1 + 2x_2 - 3x_3 + + x_1x_2 + x_2x_1 + 2x_1x_3 + 2x_3x_1 + 0x_3x_2 + 0x_2x_3$

Let $A$ be matrix form of given Quadratic Form (Q.F.) Then, $A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & -3 \end{bmatrix}$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $X' = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$

$\Rightarrow X'A = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 4x_1 + x_2 + 2x_3 & x_1 + 2x_2 + 0x_3 & 2x_1 + 0x_2 - x_3 \end{bmatrix}$

$\therefore X'AX = \begin{bmatrix} 4x_1 + x_2 + 2x_3 & x_1 + 2x_2 + 0x_3 & 2x_1 + 0x_2 - x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= (4x_1 + x_2 + 2x_3)x_1 + (x_1 + 2x_2 + 0x_3)x_2 + (2x_1 + 0x_2 - x_3)x_3$

$\therefore X'AX = 4x_1^2 + 2x_2^2 - 3x_3^2 + 2x_1x_2 + 4x_1x_3$. Hence the solution.
**Problem #4** Write down the matrix of the Quadratic Form and verify that they can be written as Matrix products $X^T A X$. $x_1^2 - 2 x_2^2 + 4 x_3^2 - 4 x_4^2 - 2x_1 x_2 + 3 x_1 x_4 + 4 x_2 x_3 - 5 x_3 x_4$

**Solution:** Let the given Quadratic Form $x_1^2 - 2 x_2^2 + 4 x_3^2 - 4 x_4^2 - 2x_1 x_2 + 3 x_1 x_4 + 4 x_2 x_3 - 5 x_3 x_4$ can be written as $x_1 x_1 - 2 x_2 x_2 + 2 x_3 x_3 + 2x_3 x_3 - 4 x_4 x_4 - x_1 x_2 + 3/2 x_1 x_4 + 3/2 x_4 x_1 + 2 x_2 x_3 + 2 x_2 x_3 -5/2 x_3 x_4 -5/2 x_3 x_4$.

Let $A$ be matrix form of given Quadratic Form (Q.F.) Then, $A = \begin{bmatrix} 1 & -1 & 0 & \frac{3}{2} \\ -1 & -2 & 2 & 0 \\ 0 & 2 & 4 & -\frac{5}{2} \\ \frac{3}{2} & 0 & -\frac{5}{2} & -4 \end{bmatrix}$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ then $X' = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}$

$\Rightarrow X'AX = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & \frac{3}{2} \\ -1 & -2 & 2 & 0 \\ 0 & 2 & 4 & -\frac{5}{2} \\ \frac{3}{2} & 0 & -\frac{5}{2} & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$= x_1^2 - 2 x_2^2 + 4 x_3^2 - 4 x_4^2 - 2x_1 x_2 + 3 x_1 x_4 + 4 x_2 x_3 - 5 x_3 x_4$

$\therefore X'AX = x_1^2 - 2 x_2^2 + 4 x_3^2 - 4 x_4^2 - 2x_1 x_2 + 3 x_1 x_4 + 4 x_2 x_3 - 5 x_3 x_4$

Hence the solution.

**Problem #5** Write down the matrix of the Quadratic Form and verify that they can be written as Matrix products $X^T A X$. $x^2 + 2 y^2 + 3 z^2 + 4 xy + 5 yz + 6xz$.

Try Ur self
Lakireddy Bali Reddy College of Engineering, Mylavaram (Autonomous)

B.Tech I Year (II-Semester) May/June 2014. T 264- Numerical Methods

UNIT – II Quadratic Forms Faculty Name: N V Nagendram

Problems Vs Answers for self Exercise Purpose Tutorial-2

**Problem #1** Obtain the Matrix corr. to Quadratic Form \( x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6zx \)?

[Ans. \( A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix} \) and order of \( A = o(A) = 3 \)]

**Problem #2** Obtain the Matrix corr. to Q. F. \( ax^2 + by^2 + cz^2 + 2gxy + 2fyz + 2hxy \)?

[Ans. \( A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \)]

**Problem #3** Obtain the Matrix corr. to Q. F. \( a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1 \)?

[Ans. \( A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \)]

**Problem #4** Obtain the Matrix corr. to Q. F. \( x_1^2 - 2x_2^2 + 4x_3^2 - 4x_4^2 - 2x_1x_2 + 3x_1x_4 + 4x_2x_3 - 5x_3x_4 \)?

[Ans. \( A = \begin{bmatrix} 1 & -1 & 0 & 3/2 \\ -1 & -2 & 2 & 0 \\ 0 & 2 & 4 & -5/2 \\ 3/2 & 0 & -5/2 & -4 \end{bmatrix} \)]

**Problem #5** Obtain the Matrix corr. to Q. F. \( d_1x_1^2 + d_2x_2^2 + d_3x_3^2 + d_4x_4^2 + d_5x_5^2 \)?

[Ans. \( A = \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 \\ 0 & 0 & 0 & d_4 & 0 \\ 0 & 0 & 0 & 0 & d_5 \end{bmatrix} \)]
Problem #6 Determine a non-singular Matrix P such that \( P'AP = \text{Diagonal Matrix} \) where

\[
A = \begin{bmatrix}
0 & 1 & 2 \\
1 & 0 & 3 \\
2 & 3 & 0
\end{bmatrix}
\]

Solution: Let \( A = \text{IAI} \Rightarrow \)

\[
\begin{bmatrix}
0 & 1 & 2 \\
1 & 0 & 3 \\
2 & 3 & 0
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
R_1 \rightarrow R_1 + R_2 \Rightarrow \begin{bmatrix}
1 & 1 & 5 \\
1 & 0 & 3 \\
2 & 3 & 0
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
C_1 \rightarrow C_1 + C_2 \Rightarrow \begin{bmatrix}
2 & 1 & 5 \\
1 & 0 & 3 \\
5 & 3 & 0
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
R_2 \rightarrow R_2 - \frac{1}{2}R_1 ; R_3 \rightarrow R_3 - \frac{5}{2}R_1 \Rightarrow \begin{bmatrix}
2 & 1 & 5 \\
0 & -\frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{2} & -\frac{25}{2}
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 0 \\
-\frac{1}{2} & 0 \\
-5 & 2 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
C_2 \rightarrow C_2 - \frac{1}{2}C_1 ; C_3 \rightarrow C_3 - \frac{5}{2}C_1 \Rightarrow \begin{bmatrix}
2 & 0 & 0 \\
0 & -\frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{2} & -\frac{25}{2}
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 0 \\
-\frac{1}{2} & 0 \\
-5 & 2 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
R_3 \rightarrow R_3 + R_1 ; C_3 \rightarrow C_3 + C_1 \Rightarrow \begin{bmatrix}
2 & 0 & 0 \\
0 & -\frac{1}{2} & 0 \\
0 & 0 & -12
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 0 \\
-\frac{1}{2} & 0 \\
-3 & 2 & 1
\end{bmatrix} \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

It is observed that L.H.S matrix is a Diagonal Matrix and there is a non-singular matrix

\[
P = \begin{bmatrix}
1 & -\frac{1}{2} & -3 \\
1 & \frac{1}{2} & -2 \\
0 & 0 & 1
\end{bmatrix}
\]

such that \( P'AP = D \).

Hence the required solution.
**Problem #7** Determine a non-singular matrix $P$ that is $|P| \neq 0$ such that $P'AP$ is a Diagonal Matrix where $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & -3 \end{bmatrix}$ interpret the result in terms of Quadratic Form?

**Solution:** Let $A = IAI \Rightarrow \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Reduce matrix $A$ to Diagonal Matrix by congruent operations on R.H.S IAI, Elementary row pre-factor on I and Elementary column post-factor on I we get as below :

$R_2 \rightarrow R_2 + (1/3)R_1$; $C_2 \rightarrow C_2 + (1/3)C_1$; and $R_3 \rightarrow R_3 - (1/3)R_1$; $C_3 \rightarrow C_3 - (1/3)C_1 \Rightarrow$

$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & -1/3 \\ 0 & -1/3 & 7/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_3 \rightarrow R_3 + (1/7)R_2$; $C_3 \rightarrow C_3 + (1/7)C_2 \Rightarrow$

$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 16/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/7 & 1/7 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix}$

On observation we obtain a non-singular matrix $P = \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix}$ such that $P'AP = D$

Where $D = \text{Diag.} [6 \ 7/3 \ 16/3 ]$.

Quadratic form corresponding to matrix $A = X'AX$

\[= 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + x_3x_1 \]

............ equation (1)

Therefore, the on-singular Transformation corresponding to $P$ is $X = PY$ i.e.,

\[X = PY \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \]

Equivalent to write in the form of linear system of equations

\[x_1 = y_1 + \frac{1}{3}y_2 - \frac{2}{7}y_3; \ x_2 = y_2 + \frac{1}{7}y_3; \ x_3 = y_3, \ldots \ldots \ldots \text{ equation (2)} \]

the transformation (2) reduce to Quadratic Form (1) to Diag. Form

\[X'P'AP = 6y_1^2 + 7/3y_2^2 + 16/7y_3^2 \]

Rank of Quadratic Form $= \rho \ (Q.F) = X'AX = 3 = \text{No. of positive square terms in Q.F.}$

Signature $= \text{sign} = \text{No. Of positive terms} - \text{No. Of negative terms} = 3 - 0 = 3$.

Hence the solution.
Definition: linear Transformation: Suppose \( V_n \) is a vector space of all ordered \( n \)-tuples of the elements of a field \( F \) and let the vectors in \( V_n \) be written as column vectors. Let \( P \) be a matrix of order \( n \) over field \( F \).

If \( Y = [y_1 \ y_2 \ y_3 \ldots \ y_n]^T \) is a vector in \( V_n \) (\( F \)). Then \( Y \) is a matrix of type \( n \times 1 \). Obviously, \( P_{nxn} Y_{nx1} = [PY]_{nx1} \) is a matrix of type \( nx1 \) is also a vector in \( V_n \). zero \( PY = X = [x_1 \ x_2 \ldots x_n]^T \)

Note: \( P(aY_1 + bY_2) = aP(Y_1) + bP(Y_2) \) is Linear Transformation.

\( P \) is one – one : \( PY_1 = PY_2 \) implies \( Y_1 = Y_2 \)
\( P \) is onto : Let \( z \in V_n \) then \( P^{-1}(Z) \in V_n \) so that \( P(P^{-1}(z)) = (PP^{-1})(z) = Iz = z \).
Therefore, \( P \) is onto.

Note: If \( PY = X \) is non-singular, then \( PY = O \Leftrightarrow Y = O \).
If \( Y = O \Rightarrow clearly PY = O \)
\[ \Leftrightarrow PY = O \Rightarrow P^{-1}(PY) = P^{-1}(O) = O \Rightarrow Y = O \text{ since } P \neq O. \]

Definition: Congruence of Matrices: A square matrix \( B \) of order \( n \) over a field \( F \) is said to be congruent to another square matrix \( A \) of order \( n \) over a field \( F \), if there exists a non-singular matrix \( P \) over a field \( F \) such that \( B = P^TAP \).

Note: The relation of congruence of matrices is an equivalence relation in the set of all \( nxn \) matrices over a field \( F \).

(i) \( A = I^TAI \)  \( \Rightarrow A = P^TBP \)
\[ \Rightarrow (P^{-1})^TAP^{-1} = B \]
\[ \Rightarrow B \equiv A. \]

(ii) \( A = P^TBP, B = Q^TCQ \)
\[ = P^TBP = P(Q^TCQ)P = (PQ)^T(CQP) = (QP)^T(CQP) \Rightarrow A \equiv C. \]
Therefore, congruence of matrices relation is equivalence.

Note: Every matrix congruent to a symmetric matrix is a symmetric matrix.

Note: \( B = (E'_s \ldots E'_2 \ E'_1)A(E_1 \ E_2 \ldots \ E_s) = P'AP \) where \( P = E_1 \ E_2 \ldots \ E_{s-1} \ E_s \) non-singular matrices is Matrix. Thus, the \( B \) obtained by \( A \) by subjecting \( A \) to a “finite chain of congruent operations is congruent to \( B \) i.e., \( B \equiv A \Rightarrow B = P'AP \).
Congruence of Quadratic Forms or Equivalence of Quadratic Forms:

**Definition:** Two Quadratic Form’s $X^TAX$, $Y^TBY$ over a field $F$ are said to be congruent or equivalent over field $F$ if their respective matrices $A$, $B$ are congruent over field $F$. Thus $X^TAX \cong Y^TBY$ if there exists a non-singular matrix $P(F)$ such that $P^TAP = B$. Quadratic forms is also congruent and so equivalence relation.

**Definition: Equivalence of real Quadratic Forms**
Two real Quadratic Forms $X^TAX$, $Y^TBY$ re said to be Real equivalent Orthogonally equivalent or complex equivalent according as there exists a non-singular Real orthogonal or a non-singular complex matrix $P$ such that $B = P^TAP$.

**Definition: Linear Transformation of Quadratic Form:**
Consider a Quadratic Form (Q.F.) $X^TAX$ (1) and a non-singular linear transformation $X = PY$...(2). So that $P$ is a non-singular matrix. Putting $X = PY$ in (1) we get,

$$X^TAX = (PY)^TAPY = Y^TBY \implies X^TAX = Y^TBY \quad \text{ (since, } B = P^TBP)$$

Thus, $Y^TBY$ is a Quadratic Form it is called a linear transformation of the form $X^TAX$ by the non-singular matrix $P$.

The matrix of Quadratic Form $Y^TBY$ is $B = P^TAP$.

$\therefore$ The Quadratic Form $Y^TBY \cong X^TAX$ where $\cong$ is congruent in relation

**Note:** The ranges of values of two congruent Q.Fs are the same

$$p = \phi = X^TAX \implies \psi = Y^TBY = p.$$ 

**Definition: Congruent reduction of a symmetric matrix:** If $A$ be any $n$-rowed non-zero symmetric matrix of rank “$r$” over a field F. Then there exists a $n$-rowed non-singular matrix $P$ over a field $F$ such that $P^TAP = \begin{bmatrix} A_1 & 0 \\ 0 & O \end{bmatrix}$ where $A_1$ is a non-singular diagonal matrix of order “$r$” over a field $F$ and each $O$ is a null matrix of suitable size.

**Definition: rank of Quadratic form $\rho(Q.F)$:**
Let $X^TAX$ be a Quadratic Form over a field $F$, the rank of the matrix $A$ is called the rank of the Q.F $X^TAX$.

If $\rho(X^TAX) = r$ then there exists $P$ such that $|P| \neq O$ reduce to the form $X^TAX$ to a sum of ‘$r$’ square terms.
Reduction of real Quadratic Form (Q.F.):

If A is any n-rowed real symmetric matrix of rank r, then there exists a real non-singular matrix P such that $P^TAP = \text{diag.} \ [1,1,\ldots,1,-1,-1,\ldots,-1,0,0,0,\ldots,0]$. So that 1, appears p times, -1 appears $(r-p)$ times.

Definition: canonical Form(C.F.) or Normal Form(N.F.) of a real Quadratic Form:

If $X^TAX$ is a real Quadratic Form in n-variables, then there exists a real non-singular linear transformation $X = PY$ which transforms $X^TAX$ to the form represented as $X^TAX = y_1^2 + y_2^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_r^2$.

New form the given Quadratic Form (Q.F.) has been expressed as a sum and difference of the squares of new variables latter expression is called “ Canonical Form(C.F.) or Normal Form(N.F.) of the given Quadratic Form.

The number of positive square terms in any two Normal reductions of Real Quadratic Form is the same i.e., $X = PY$ ; $X = PZ \Rightarrow y_1^2 + y_2^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_r^2 \quad \text{equation(1)}$

$z_1^2 + z_2^2 + \ldots + z_p^2 - z_{p+1}^2 - \ldots - z_r^2 \quad \text{equation(2)}$

$\Rightarrow p = q$

Definition: Let $y_1^2 + y_2^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_r^2$ be a normal form (N.F.) of a Real Quadratic Form(Q.F.) $X'AX$ of rank “r”. The number “p” of positive terms in a Normal Form(N.F.) of $X'AX$ is called the index of the Quadratic Form(Q.F). The express of the number of positive square terms over the number of negative square terms in a Normal Form(N.F.) of $X'AX$ that is $p - (r - p) = 2p - r$ is called a signature of the Q.F. and is usually denoted by s. Therefore, $s = 2p - r$. 
Problem #1 Reduce \( 2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2 \) Quadratic Forms in three variables to Real Canonical Form (C.F.) and find its rank and signature. Also write the linear transformation which brings to Normal Form.

Problem #2 Reduce \( x_1^2 - 2y^2 + 3z^2 - 4yz + 6zx \) Quadratic Forms in three variables to Real Canonical Form (C.F.) and find its rank and signature. Also write the linear transformation which brings to Normal Form.

Problem #3 Reduce \( 6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_2x_3 + 18x_3x_1 + 4x_1x_2 \) Quadratic Forms in three variables to Real Canonical Form (C.F.) and find its rank and signature. Also write the linear transformation which brings to Normal Form.

Problem #4 Reduce \( x_1^2 + 2y^2 + 2z^2 - 2xy - 2yz + zx \) Quadratic Forms in three variables to Real Canonical Form (C.F.) and find its rank and signature. Also write the linear transformation which brings to Normal Form.

Problem #5 Reduce \( x_1^2 + 4y^2 + 9z^2 + t^2 - 12yz + 6zx - 4xy - 2xt - 6zt \) Quadratic Forms in four variables to Real Canonical Form (C.F.) and find its rank and signature. Also write the linear transformation which brings to Normal Form.

Problem #6 Find an orthogonal Matrix \( P \) that will Diagonalizes the real symmetric matrix

\[
A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}
\]

interpret the result in Quadratic Form (Q.F.).

---

[Hint. A = IAI, on applying Congruence operations (1) \( R_2 \rightarrow R_2 - 3R_1; \ C_2 \rightarrow C_2 - 3C_1; \) \( R_3 \rightarrow R_3 + R_1;\ C_3 \rightarrow C_3 + C_1 \) (2) \( R_3 \rightarrow R_3 + \frac{2}{17} R_2; \ C_3 \rightarrow C_3 + \frac{2}{17} C_2 \) & \( \frac{17}{\sqrt{17}} C_3 \). N.F. is \( y_1^2 - y_2^2 - y_3^2 \). Rank = \( p(Q.F.) = 3 \); signature of Q.F. = positive Sq. terms – negative sq. Terms = 1 – 2 = -1. Index is no. Of positive terms in N.F. = 1. Required Solution is \( x_1 = ay_1 - 3b y_2 + \frac{11}{17} cy_3; x_2 = b y_2 + \frac{21}{17} cy_3; x_3 = c y_3 \) Hence the solution.]
Problem #7 Reduce the Quadratic form (Q.F.) $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ to Canonical Form (C.F.) by Orthogonal Transformation (O.T.)

Solution: Let us write Quadratic form (Q.F.) $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3 + 0 x_1 x_2 + 0 x_3 x_1$ into corresponding symmetric matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Characterisitic equation $| A - \lambda I | = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(\lambda-2)(\lambda-4) = 0$

$\Rightarrow \lambda = 1, 2, 4$

Case i: $\lambda = 1$

$$\begin{bmatrix} 1-1 & 0 & 0 \\ 0 & 3-1 & -1 \\ 0 & -1 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

0.$x_1 - 0.x_2 + 0.x_3 = 0$

2.$x_2 - x_3 = 0$

$$x_1/4-1 = x_2 / 0 = x_3 / 0 = k$$ implies

$$x_1 = 1; x_2 = x_3 = 0$$

E.Vs $x_1 = 0; x_2 = x_3 = k$

$$X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Normalized vector$=$

$$\frac{x}{\sqrt{\sum x^2}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Case ii: $\lambda = 1$

$$\begin{bmatrix} 1-2 & 0 & 0 \\ 0 & 3-2 & -1 \\ 0 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

0.$x_1 - 0.x_2 + 0.x_3 = 0$

$$x_2 - 3.x_3 = 0$$ implies

$$x_2 = k, x_3 = k, x_1 = 0$$

E.Vs $x_1 = 0; x_2 = x_3 = k$

$$X = \begin{bmatrix} 0 \\ 1/k \\ 1 \end{bmatrix}$$

Normalized vector$=$

$$\frac{x}{\sqrt{\sum x^2}} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Case iii: $\lambda = 4$

$$\begin{bmatrix} 1-4 & 0 & 0 \\ 0 & 3-4 & -1 \\ 0 & -1 & 3-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

-3.$x_1 = 0$ and

- $x_2 - x_3 = 0$

$$x_1 = 0, x_2 = k, x_3 = k$$ implies

E.Vs $x_1 = 0; x_2 = k, x_3 = k$

$$X = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Normalized vector$=$

$$\frac{x}{\sqrt{\sum x^2}} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

P is orthogonal $P^T = P^{-1}$
\[ P^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \] such that Diagonalization \( D = P^T A P \)

\[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} A \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \]

Canonical Form C.F. = \( y_1^2 + 2y_2^2 + 4y_3^2 \)

\[ X = PY \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \]

Implies \( x_1 = y_1 \); \( x_2 = \frac{y_2}{\sqrt{2}} \); \( x_3 = \frac{y_2 - y_3}{\sqrt{2}} \)

Hence the required solution.
Problem #8 Reduce the Quadratic form (Q.F.) \(2x^2 + 2y^2 + 2z^2 - 2xy - 2yz + 2zx\) to Canonical Form (C.F.) by Orthogonal Transformation (O.T.)

Solution: Let us write Quadratic form (Q.F.) \(2x^2 + 2y^2 + 2z^2 - 2xy - 2yz + 2zx\) into corresponding symmetric matrix

\[
A = \begin{bmatrix}
2 & -1 & 1 \\
-1 & 2 & -1 \\
1 & -1 & 2
\end{bmatrix}
\]

Characteristic equation \(\det(A - \lambda I) = 0\)

\[
\begin{vmatrix}
2 - \lambda & -1 & 1 \\
-1 & 2 - \lambda & -1 \\
1 & -1 & 2 - \lambda
\end{vmatrix} = 0
\]

\Rightarrow (2-\lambda)[(2-\lambda)^2 - 1] + 1[-1(2-\lambda) - (2-\lambda)] + 1[-(2-\lambda)] = 0
\Rightarrow - \lambda^3 + 6 \lambda^2 - 9 \lambda + 4 = 0 \text{ implies } \lambda = 1, 1, 4

Case i: \(\lambda = 1\)

\[
\begin{bmatrix}
2-1 & -1 & 1 \\
-1 & 2-1 & -1 \\
1 & -1 & 2-1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[x_1 - x_2 + x_3 = 0\]
\[-x_1 + x_2 - x_3 = 0\]
\[x_1 - x_2 + x_3 = 0\]
implies
\[x_1 = 0 \text{ implies } x_2 = x_3\]
\[x_3 = 1 \text{ implies } x_2 = 1\]

\[
X = \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}
\]

Normalized vector = \[
\sqrt{\sum_i x_i^2} = \begin{bmatrix}
0 \\
1/\sqrt{2} \\
1/\sqrt{2}
\end{bmatrix}
\]

Case ii: \(\lambda = 1\)

On solving similarly

Case iii: \(\lambda = 4\)

On solving we get

\[
X = \begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}
\]

\[
X = \begin{bmatrix}
-1 \\
1 \\
1
\end{bmatrix}
\]

\[
\sqrt{\sum_i x_i^2} = \begin{bmatrix}
1/\sqrt{6} \\
1/\sqrt{6} \\
1/\sqrt{6}
\end{bmatrix}
\]

\[
\sqrt{\sum_i x_i^2} = \begin{bmatrix}
1/\sqrt{3} \\
1/\sqrt{3} \\
1/\sqrt{3}
\end{bmatrix}
\]

\[
P = \begin{bmatrix}
0 & 1/\sqrt{6} & 1/\sqrt{3} \\
1/\sqrt{2} & 2/\sqrt{6} & -1/\sqrt{3} \\
1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3}
\end{bmatrix}
\]

\(P\) is orthogonal \(P^T = P^{-1}\)
\[ \mathbf{P}^T = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \] such that Diagonalization \( \mathbf{D} = \mathbf{P}^T \mathbf{A} \mathbf{P} \)

\[
\begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \mathbf{A} \begin{bmatrix} 0 & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 2/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}
\]

**Canonical Form C.F.** \( y_1^2 + y_2^2 + 4y_3^2 \)

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 2/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}
\]

Implies \( x_1 = \frac{y_1 + y_3}{\sqrt{6}} \); \( x_2 = \frac{y_1 + 2y_2 - y_3}{\sqrt{2}} \); \( x_3 = \frac{y_1 + y_2 + y_3}{\sqrt{3}} \)

Hence the required solution.
01. State Sylvester law

If \( P(A) = C_0 A^n + C_1 A^{n-1} + C_2 A^{n-2} + \ldots + C_{n-1} A + C_n \) I and \( f(\lambda) \mid A - \lambda I \) and matrix

\[
[f(\lambda)] = \text{Adj. Matrix of } [\lambda I - A]
\]

then

\[
P(A) = \sum_{i=1}^{n} P(\lambda_i) Z(\lambda_i)
\]

02. If \( A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \) then \( A^{50} = \ldots \) [Ans. \( \begin{bmatrix} 1 & 0 \\ 0 & 3^{50} \end{bmatrix} \)]

03. If \( A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \) then \( A^{100} = \ldots \) [Ans. \( \begin{bmatrix} 1 & 0 \\ 0 & 3^{100} \end{bmatrix} \)]

04. If \( A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \) then \( e^A = \ldots \) [Ans. \( \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix} \)]

05. The relation \( Y = AX \) represents the transformation known as............................ [Ans. Linear Transformation]

06. The linear transformation \( Y = AX \) is said to be orthogonal if ......................... [Ans. \( \sum_{i=1}^{n} x_i^2 \approx \sum_{i=1}^{n} y_i^2 \)]

07. If \( \lambda_1, \lambda_2, \lambda_3 \) are the eigenvalues of the matrix of a quadratic form, then canonical form is..

[Ans. \( \sum_{i=1}^{3} \lambda_i y_i^2 \)]

08. The index of Quadratic Form (Q. F.) is ...................... [Ans. No of + square terms in the canonical Form (C.F.)]

09. Signature of a Quadratic Form is................................................................. [Ans. The difference between number of + square terms & - square terms of N.F or C.F.]

10. A necessary and sufficient condition for a real quadratic form \( X'AX \) to be positive definite is, that the leading principal minors of \( A \) are all ................................................................. [Ans. positive]
Some Important Definitions:

**Index:** Canonical Form (C.F.) of Quadratic Form (Q.F.) $Q = X'AX$ is $Y'DY$ or $\lambda_1 y_1^2 + \lambda_1 y_1^2 + \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$ is orthogonal Transformation (O.T.) $X = PY$ where $P$ is Modal Matrix or Transformation of Matrix, $D$ is spectral Matrix or Diagonal Matrix Diag. $[\lambda_1 \lambda_2 \lambda_3 \lambda_4 \cdots \lambda_n]$ and rank of $A = \rho(A) = r$, $n =$ number of variables $x_i$ in Quadratic Form (Q.F.)

Then we can define $s =$ index of Quadratic Form (Q.F.) $=$ number of positive square terms in Canonical Form (C.F.).

**Signature of Quadratic Form (Q.F.)** $=$ Number of positive square terms $-$ Number of negative square terms in Canonical Form C.F.

**Definiteness:** $| Q | \neq 0$, $Q = X'^TAX$ and $| A | \neq 0$ is definiteness.

**Positive Definiteness:** if $\rho(A) = r = \text{index} (A) = s = n$ or All $\lambda_i$ of $A$ is positive.

**Negative Definiteness:** index $(A) = 0$ ; $\rho(A) = r = \text{index} (A) = n$, $s = 0$ or All $\lambda_i$ of $A$ is $-$ ve.

**Positive semi definiteness:** $\rho(A) = r = \text{index} (A) \Rightarrow s = r < n$ $(|A| \neq 0)$ or All $\lambda_i \geq 0$ of $A$ and for some $i$, is $\lambda_i = 0$.

**Negative semi definiteness:** $\rho(A) = r = \text{index} (A) \Rightarrow s = r < n$ $(|A| \neq 0)$ or All $\lambda_i \leq 0$ of $A$ and for some $i$, is $\lambda_i = 0$.

**Indefiniteness:** for all $\lambda_i(A)$, some of $\lambda_i$ 's $\lambda_i =$ positive, some of $\lambda_i$ 's $\lambda_i =$ negative.
Introduction: In many engineering problems, it is required to find the solution of the equation of the form \( f(x) = 0 \) where \( f(x) = 0 \) may be algebraic or transcendental equation of higher order.

In this chapter, various numerical approximation methods are used to solve such algebraic and transcendental equations.

The limitations of analytical methods led the engineers and scientists to evolve graphical and numerical methods.

Numerical methods often a repetitive nature. These consist in repeated execution of the same process where at each step the result of the proceeding step is used. This is known as “Iteration Process” and is repeated till the result obtained to desired degree of accuracy.

In this, we shall discuss some numerical methods for the solution of algebraic and transcendental equations and simultaneous linear and non-linear equations. We can closed the chapter by describing an iterative method for the solution of Eigen value Problem (E V P).

### Solution of Algebraic and Transcendental Equations

**Type**

- BISECTION METHOD – I
- Method of False Position or Regula Falsi Method – II
- Newton Raphson Method - III
Definition: Polynomial: An expression of the form \( f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \) is called a polynomial in \( x \) of degree \( n \), provided \( a_0 \neq 0 \) where \( a_0, a_1, a_2, a_3, \ldots, a_n \) are constants, real or complex.

Definition: Algebraic Equation: A polynomial in \( x \) of degree \( n \), when equated to zero i.e., \( f(x) = 0 \) is called an algebraic equation of degree \( n \).

Definition: Transcendental Equation: If the polynomial \( f(x) \) involves the functions of the form such as trigonometric, logarithmic, exponential etc., then \( f(x) = 0 \) is called a transcendental equation.

Properties of Polynomial Equation:
(i) Every polynomial equation of degree \( n \) has exactly \( n \) roots, real or complex is known as **Fundamental theorem of algebra**.

(ii) Every polynomial equation \( f(x) = 0 \) of degree \( n \geq 1 \) has at least one root, real or imaginary.

(iii) Imaginary roots occur in pairs i.e., if \( \alpha + i\beta \) is a root, \( \alpha - i\beta \) is also a root of the equation and so every equation of the odd degree has at least one real root.

(iv) Irrational roots occur in pairs i.e., if \( a + \sqrt{b} \) is a root of an equation, \( a - \sqrt{b} \) must be its root.

(v) If \( f(x) \) is continuous in the closed interval \([a, b]\) and \( f(a) \) and \( f(b) \) are of opposite signs, then the equation \( f(x) = 0 \) has at least one root between \( x = a \) and \( x = b \) is known as **Intermediate value theorem**.

(vi) The number of positive roots of a polynomial equation \( f(x) = 0 \) with real coefficients cannot exceed the number of changes of sign of the coefficients in \( f( x ) \) and the number of negative roots cannot exceed the number of changes of sign of the coefficients in \( f( - x ) \), is known as **Descarte’s rule of signs**.
This method is based on repeated application of the intermediate value theorem.

Let the function $f(x)$ be continuous in a closed interval $[a, b]$.

If $f(a)$ and $f(b)$ are of opposite signs, at least one real root lies between $a$ and $b$.

For definiteness, let $f(a)$ be negative and $f(b)$ be positive so that $f(a)f(b) < 0$. Bisection method is used to find the root between $a$ and $b$.

(i) The approximation of the required root is given by
$$x_1 = \frac{a + b}{2}.$$

(ii) If $f(x_1) = 0$ then $x_1$ is a root of the equation. If $f(x_1) \neq 0$ then the root lies between $a$ and $x_1$ provided $f(x_1)$ is positive. (or) the root lies between $x_1$ and provided $f(x_1)$ is negative.

(iii) If $f(x_1)$ is positive the second approximation to the required root is given by
$$x_2 = \frac{a + x_1}{2}.$$

(iv) If $f(x_2)$ is negative the third approximation to the required root is given by
$$x_3 = \frac{x_1 + x_3}{2}.$$

(v) Bisect the interval in which the root lies and continue the process to obtain the root to the desired level of accuracy.
Covergence of Bisection Method:

Let a function \( f(x) \) be continuous in a closed interval in \([a, b]\) and there exists a number \( c \) between \( a \) and \( b \).

Let \( f(a) \) and \( f(b) \) are of positive signs and \( x_1, x_2, x_3, \ldots \) is the sequence of mid points obtained by bi-section method then \( |c - x_n| \leq \frac{b-a}{2^n} \) for \( n \) is 1, 2, 3, \ldots and therefore the sequence \( \{x_n\} \) converges to the root \( c \) i.e., \( \lim x_n = c \) as \( n \) tends to \( \infty \).

The Bi-section Method is simple and the most reliable one. At each iteration, we bisect the interval and so only one binary digit precision can be obtained at each iteration. The convergence is very slow but definite.
Problem # 01  Solve $x^3 - 9x + 1 = 0$ for the root to three decimals between $x = 2$ and $x = 4$ by the method of interval halving?
[Ans. $2 < 2.943 < 4$ ]

Problem # 02  Find the root between 2 and 3 of the equation $x^4 - x^3 - 2x^2 - 6x - 4 = 0$ ?
[Ans. $2 < 2.7315 < 3$ ]

Problem # 03 Using Bi-section Method, find the negative root of $x^3 - 4x - 9 = 0$ ?
[Ans. $-3 < -2.711 < -2$ ]

Problem # 04 Find a real root of equation $x^3 - x - 11 = 0$ by using Bi-section method ?
[Ans. $2.25 < 2.375 < 2.5$ ]

Problem # 05 Find a real root of equation $x^3 - 5x + 3 = 0$ by using Bi-section method?
[Ans. $1 < 1.8125 < 2$ ]

Problem # 06 Find a real root of equation $x^3 - 6x - 4 = 0$ by using Bi-section method?
[Ans. $2 < 2.71875 < 3$ ]
Problem # 07 Find a real root of equation \( x \log_{10} x = 1.2 \) which lies between 2 and 3 by using Bi-section method?
[Ans. 2 < 2.6875 < 3]

Problem # 08 Find a positive root of equation \( x^3 - 4x - 9 = 0 \) by using Bi-section method in four stages?
[Ans. 2 < 2.6875 < 3]

Problem # 09 Find an approximate root of the equation \( \sin x = \frac{1}{x} \) that lies between \( x = 1 \) and \( x = 1.5 \) measured in radians by using Bi-section method carry out computation up to 7th stage?
[Ans. 1 < 1.11328125 < 1.5]

Problem #10 Find the square root of 25 given \( x_0 = 2.0 \) and \( x_1 = 7.0 \) by using Bi-section method?
[Ans. square root of 25 is 5 such that 4.5 < 5 < 7.0]

Problem #11 Find a positive root of equation \( x^3 - x - 1 = 0 \) correct to two decimal places by using Bi-section method?
[Ans. root is 1.32 such that 1 < 1.32 < 2]

Problem #12 Find a root of equation \( x^3 - 5x + 1 = 0 \) by using the Bi-section method in 5 stages?
[Ans. 0 < 0.21875 < 1]

Problem #13 Evaluate a real root of the equation \( 4 \sin x = e^x \) by using Bi-section Method?
[Ans. 0 < 0.36718 < 0.5]

Problem #14 Find the positive root of the equation \( x - \cos x = 0 \) by Bi-section Method?
[Ans. 0.5 < 0.7388 < 1.0]
Problem # 07 Find a real root of equation \( x \log_{10} x = 1.2 \) which lies between 2 and 3 by using Bi-section method?

[Ans. 2 < 2.6875 < 3 ]

Solution: Let \( f( x ) \equiv x \log_{10} x - 1.2 = 0 \)

Since \( f(2) = 2. \log_{10} 2 - 1.2 = -0.598 \) negative and

\( f(3) = 3. \log_{10} 3 - 1.2 = 0.2313 \) positive

Therefore, a root lies between 2 and 3.

\[ \therefore \text{Ist approximation to the root is } x_1 = \frac{1}{2} (2 + 3) \approx 2.5 \]

Then, \( f(x_1) = (2.5).10^{2.5} - 1.2 = -0.2053 \) is negative sign

\[ \therefore f(3) \text{ is positive and The root lies between } x_1 \text{ and } 3. \]

\[ \therefore \text{The second approximation to the root is } x_2 = \frac{1}{2} (x_1 + 3) \approx 2.75 \]

Then \( f(x_2) = (2.75). \log_{10} 2.75 - 1.2 = 0.008 \) is positive sign

\[ \therefore f(2.5) \text{ is positive and The root lies between } x_1 \text{ and } x_2. \]

\[ \therefore \text{The third approximation to the root is } x_3 = \frac{1}{2} (x_1 + x_2) \approx 2.625 \]

Then \( f(x_3) = (2.625).log_{10} 2.625 - 1.2 = -0.10 \) is negative sign

\[ \therefore f(2.75) \text{ is positive and The root lies between } x_2 \text{ and } x_3. \]

\[ \therefore \text{The fourth approximation to the root is } x_4 = \frac{1}{2} (x_2 + x_3) \approx 2.6875 \]

\[ \therefore \text{The root is 2.6875 approximately.} \]

Hence the required solution.
**Problem # 08** Find a positive root of equation \( x^3 - 4x - 9 = 0 \) by using Bi-section method in four stages?

[Ans. 2 < 2.6875 < 3 ]

**Solution:** Let \( f( x ) \equiv x^3 - 4x - 9 = 0 \)

Since \( f( 2 ) = 8 - 8 - 9 = -9 \) negative and
\( f( 3 ) = 27 - 12 - 9 = 6 \) positive
Therefore, a root lies between 2 and 3.

\[ \therefore \text{Ist approximation to the root is } x_1 = \frac{1}{2} (2 + 3) \approx 2.5 \]

Then, \( f( x_1 ) = (2.5)^3 - 4(2.5) - 9 = -3.375 \) is negative sign
\[ \therefore \text{The root lies between } x_1 \text{ and } 3. \]

\[ \therefore \text{The second approximation to the root is } x_2 = \frac{1}{2} (x_1 + 3) \approx 2.75 \]

Then \( f( x_2 ) = (2.75)^3 - 4(2.75) - 9 = 0.7969 \) is positive sign
\[ \therefore \text{The root lies between } x_1 \text{ and } x_2. \]

\[ \therefore \text{The third approximation to the root is } x_3 = \frac{1}{2} (x_1 + x_2) \approx 2.625 \]

Then \( f( x_3 ) = (2.625)^3 - 4(2.625) - 9 = -1.4121 \) is negative sign
\[ \therefore \text{The root lies between } x_2 \text{ and } x_3. \]

\[ \therefore \text{The fourth approximation to the root is } x_4 = \frac{1}{2} (x_2 + x_3) \approx 2.6875 \]

\[ \therefore \text{The root is } 2.6875 \text{ approximately.} \]

Continuing the procedure, the successive approx. are \( x_5 = 2.71875, x_6 = 2.703125, x_7 = 2.7109375, x_8 = 2.7070313, x_9 = 2.7050782. \)

Hence the required solution.
**Problem #11** Find a positive root of equation $x^3 - x - 1 = 0$ correct to two decimal places by using Bi-section method?

[Ans. root is 1.32 such that $1 < 1.32 < 2$]

**Solution:** Let $f(x) = x^3 - x - 1 = 0$

Since $f(1) = 1 - 1 - 1 = -1$ negative and $f(2) = 8 - 2 - 1 = 5$ positive

Therefore, a root lies between 1 and 2. Here $f(0) = -1 = f(1)$ and consider $f(1) = -1$ to get closer approximation between two consecutive values of $x$. Also $f(1.5) = 0.8750$

∴ The root lies between 1 and 1.5 (instead of shorten range 1 and 2).

∴ Ist approximation to the root is $x_1 = \frac{1}{2} (1 + 1.5) \approx 1.2500$

Then, $f(x_1) = (1.25)^3 - (1.25) - 1 = -0.29688$ is negative sign

∴ $f(1.5) = 0.8750$ is positive and The root lies between $x_1$ and 1.5.

∴ The second approximation to the root is $x_2 = \frac{1}{2} (x_1 + 1.5) \approx 1.375$

Then $f(x_2) = (1.375)^3 - (1.375) - 1 = 0.22461$ is positive sign

∴ $f(1.25) = -0.29688$ is negative and The root lies between $x_1$ and $x_2$.

∴ The third approximation to the root is $x_3 = \frac{1}{2} (x_1 + 1.375) \approx 1.3125$

Continuing the procedure, the successive approximations are $x_4 = 1.3438$, $x_5 = 1.3282$, $x_6 = 1.3204$, $x_7 = 1.3243$, $x_8 = 1.3263$, $x_9 = 1.3253$, $x_{10} = 1.3248$, $x_{11} = 1.32455$, $x_8 = 1.3247$.

∴ The root is 1.3247 approximately.

Hence the required solution.
Problem #14 Find the positive root of the equation \( x - \cos x = 0 \) by Bi-section Method?

**Solution:** Let \( f(x) \equiv x - \cos x = 0 \)

Since \( f(0) = 0 - 1 = -1 \) negative and
\[
\begin{align*}
  f(0.5) &= 0.5 - \cos(0.5) = -0.37758 \text{ is negative} \\
  f(1.0) &= 1.0 - \cos(1.0) = 0.4597 \text{ is positive}
\end{align*}
\]

Therefore, a root lies between 0.5 and 1.0.

\[
\therefore \text{Ist approximation to the root is } x_1 = \frac{1}{2}(0.5 + 1.0) \approx 0.75
\]

Then, \( f(x_1) = (0.75) - \cos(0.75) = 0.018311 \) is positive sign

\[
\therefore \text{The second approximation to the root is } x_2 = \frac{1}{2}(x_1 + 0.5) \approx 0.625
\]

Then \( f(x_2) = (0.625) - \cos(0.625) = -0.18596 \) is negative sign

\[
\therefore \text{The third approximation to the root is } x_3 = \frac{1}{2}(x_1 + 0.6825) \approx 0.71875
\]

Continuing the procedure, the successive approximations are \( x_4 = 0.73438, x_5 = 0.74219, x_6 = 0.73829, x_7 = 0.7402, x_8 = 0.73925, x_9 = 0.7388 \)

\[
\therefore \text{The root is 0.7388 approximately.}
\]

Hence the required solution.
METHOD – II : THE METHOD OF FLASE POSITION (REGULA FALSI METHOD)

This is the oldest method of finding the real root of an equation \( f(x) = 0 \) and closely resembles the Bi-section Method.

Let us choose two points \( x_0 \) and \( x_1 \) such that \( f(x_0) \) and \( f(x_1) \) are of opposite signs i.e., the graph of \( y = f(x) \) crosses the \( X \)-axis between these points.

This indicates that root lies between \( x_0 \) and \( x_1 \) consequently \( f(x_0) - f(x_1) < 0 \).

Equation of the chord joining two points \( A(x_0, f(x_0)) \) and \( B(x_1, f(x_1)) \) is defined as

\[
y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0) \quad \text{......equation(1)}
\]

This method consists in replacing the curve \( AB \) by means of the chord \( AB \) and taking the point of intersection of the chord with the \( x \)-axis as an approximation to the root.

So, the abscissa of the point where the chord cuts the \( x \)-axis ( \( y = 0 \) ) is given by as below

\[
x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \quad \text{......equation(2)}
\]

which is the approximation to the root. If now \( f(x_0) \) and \( f(x_2) \) are of opposite signs, then the root lies between \( x_0 \) and \( x_2 \). So replacing \( x_1 \) by \( x_2 \) in equation (2) we obtain approximation \( x_3 \).

Root lies between \( x_1 \) and \( x_2 \) and we would obtain \( x_3 \) approximately. Accordingly, this procedure is repeated till the root is found to be desired accuracy. The iteration process based on equation (1) is known as the “Method of false position” or “Regula Falsi Method”.

**False Position or Regula – Falsi Method Figure**
Lakireddy Bali Reddy College of Engineering, Mylavaram  
(Autonomous)  
B.Tech I Year  (II-Semester) May/ June 2014.  
T 264- Numerical Methods  
UNIT – II  
Quadratic Forms  
Faculty Name: N V Nagendram  
Problems on Method of False Position  
Tutorial-6

**Problem # 01** Find the smallest real root $x^2 - \log_e x - 12 = 0$ by the method of false position?  
[Ans. $3 < 3.6461 < 4$]

**Problem # 02** Find the smallest real root $x - \cos x = 0$ by the method of false position?  
[Ans. $0 < 0.7391 < 1$]

**Problem # 03** Find a smallest real root of equation $x^3 - x - 1 = 0$ by the method of false position?  
[Ans. root is 1.32 such that $1 < 1.3232 < 2$]

**Problem # 04** Find a smallest real root of equation $x^3 - 2x - 5 = 0$ by the method of false position?  
[Ans. root is 1.32 such that $2 < 2.0945 < 3$]
Problems and Solutions on Method False Position (M F P):
(REGULA - FALSI METHOD)

Problem # 01 Find a real root of the equation \( x^3 - 2x - 5 = 0 \) by the method of false position correct to three decimal places?

Solution: Let \( f(x) \equiv x^3 - 2x - 5 = 0 \)

\[
\begin{align*}
f(2) &= 8 - 4 - 5 = -1 \text{ is negative sign} \\
f(3) &= 27 - 6 - 5x_1 = 16 \text{ is positive sign}
\end{align*}
\]

\( \therefore \) A root between 2 and 3.

\( \because \) Taking \( x_0 = 2 \), \( x_1 = 3 \) so \( f(x_0) = -1 \)

\[
\begin{align*}
f(x_1) &= 16 \text{ in this method of false position} \\
\text{we get, } x_2 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \frac{1}{7} = 2.0588
\end{align*}
\]

\[
\begin{align*}
f(x_2) &= f(2.0588) = (2.0588)^3 - 2(2.0588) - 2(2.0588) - 5 = -0.3908 \text{ so, } 2 < -0.3908 < 3
\end{align*}
\]

\( \therefore \) Taking \( x_0 = 2.0588 \), \( x_1 = 3 \) so \( f(x_0) = -0.3908 \)

\[
\begin{align*}
f(x_1) &= 16 \text{ in this method of false position} \\
\text{we get, } x_3 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2.0588 - \frac{0.94121}{16.3908} (-0.3908) = 2.0813
\end{align*}
\]

On repeating this process,

The successive approximations are \( x_4 = 2.0862, x_5 = 2.0915, x_6 = 2.0934, x_7 = 2.0941, x_8 = 0.0943 \) etc.,

\( \therefore \) The root is 2.094 correct to three decimals. Hence the solution.
**Problem #02** Find the root of the equation \( x^3 - 3x + 4 = 0 \) correct to three decimals by Method of False Position (M.F.P.).

**Solution:** Let \( f(x) \equiv x^3 - 3x + 4 = 0 \)

So that \( f(2) = 8 - 6 + 4 = 6 \) is positive sign

\( f(3) = 27 - 9 + 4 = 22 \) is positive sign

\( \therefore \) A root between 2 and 3.

\( \therefore \) Taking \( x_0 = 2, \ x_1 = 3 \) so \( f(x_0) = 6 \)

\( \quad f(x_1) = 22 \) in this method of false position (M.F.P.)

we get, \( x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 - \frac{3 - 2}{22 - 6}.(6) \approx 2 - \frac{6}{16} \approx \frac{13}{16} = 1.9375 \)

therefore, \( 2 < 1.9375 < 3 \) and \( x_2 = 1.9375 \)

\( f(x_2) = f(1.9375) = (1.9375)^3 - 3(1.9375) + 4 = 5.9185; \) so, \( 2 < 1.9375 < 3 \)

\( \therefore \) Taking \( x_0 = 1.9375, \ x_1 = 3 \) so \( f(x_0) = 6 \)

\( \quad f(x_1) = 22 \) in this method of false position

we get, \( x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 1.9375 - \frac{3 - 1.9375}{5.9185 - 22}.(22) \approx 2.0035 \)

Therefore, \( 2 < 2.0035 < 3 \)

On repeating this process,

\( \therefore \) The root is \( 2.0035 \) correct to four decimals. Hence the solution.
Problems and Solutions on Regula Falsi Method (RFM):
(FALSE POSITION METHOD)

**Problem # 01** Find a real root of the equation \( x \log_{10} x = 1.2 \) by Regula-Falsi method correct to 4 decimals?

**Problem # 02** Find a real root of the equation \( x.e^x = \cos x \) by the Regula-Falsi method correct to 4 decimals?

**Problem # 03** Find a real root of the equation \( x.e^x - 2 = 0 \) by the Regula-Falsi method correct to 3 decimals?  
[Ans. 0.853]

**Problem # 04** Find a real root of the equation \( 2x.\log x = 6 \) by the Regula-Falsi method correct to 4 decimals?  
[Ans. 3.257]

**Problem # 05** Find a real root of the equation \( x.e^x - \sin x = 0 \) by the Regula-Falsi method correct to 4 decimals?  
[Ans.-0.134]
**Problem # 01** Find a real root of the equation $x \log_{10} x = 1.2$ by Regula-Falsi method correct to 4 decimals?

**Solution:**
Let $f(x) = x \log_{10} x - 1.2$

So that:
- $f(1) = 1 \log_{10} 1 - 1.2$ is negative sign
- $f(2) = 2 \log_{10} 2 - 1.2$ is negative sign
- $f(3) = 3 \log_{10} 3 - 1.2$ is positive sign

Therefore, a root lies $2 < \text{root} < 3$

∴ Taking $x_0 = 2$, $x_1 = 3$ so $f(x_0) = -0.59794$
$f(x_1) = 0.23136$ in this method of false position

we get, $x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 - \frac{3 - 2}{0.23136 + 0.59794} (-0.59794)$

$x_2 \approx 2.72102$

⇒ $f(x_2) = f(2.72102) = 2.72102 \log_{10} 2.72102 - 1.2 = -0.01709$

And $f(x_1) = 0.23136$ in (1) we get

$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2.72102 + \frac{0.27898}{0.23136 + 0.01709} (0.01709)$

$x_3 \approx 2.74021$

On repeating this process the successive approximations are

$x_4 = 2.74024$ ; $x_5 = 2.74063$ so on.

∴ the root is 2.7406 correct to 4 decimals.

Hence the required solution.
Problem # 02. Find a real root of the equation $x.e^x = \cos x$ by the Regula-Falsi method correct to 4 decimals?

Solution: Let $f(x) = x.e^x - \cos x = \cos x - x.e^x = 0$

So that $f(0) = \cos 0 - 0.e^0 = 1$ is positive sign

$f(1) = \cos 1 - 1.e^1 = \cos 1 - e = -2.17798$ is -ve sign

Therefore, a root lies $1 < \text{root} < 1$

Taking $x_0 = 0$, $x_1 = 1$ so $f(x_0) = 1$

$f(x_1) = -2.17798$ in this method of Regula falsi-method we get,

$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 0 - \frac{1-0}{-2.17798-1}(1) \quad \ldots \ldots (1)$

$\approx 0 + \frac{1}{3.17798}(1)$

$x_2 \approx 0.31467$

$\Rightarrow f(x_2) = f(0.34167) = \cos(0.34167) - 0.34167.e^{0.34167}$

$= 0.15900 \times 0.36087 \times 0.51987$

$\therefore f(0.34167) = 0.51987 \quad \text{Hence} \quad 0.34167 < \text{root} < 1$

And $f(x_1) = -2.17798$ in (1) we get

$\therefore$ Taking $x_0 = 0.34167, x_1 = 1$ so $f(x_0) = 0.51987$

$f(x_1) = -2.17798$ in (1) this method of Regula falsi-method we get,

$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 0.34167 + \frac{1-0.34167}{-2.17798-0.51987}(0.51978) \quad \ldots \ldots (1)$

$\approx 0.34167 + \frac{0.68533}{0.69785}(0.51978)$

$x_3 \approx 0.44673$

$\Rightarrow f(x_3) = f(0.44673) = \cos(0.44673) - 0.44673.e^{0.44673} = 0.20356$

$\therefore f(0.44673) = 0.20356 \quad \text{Hence} \quad 0.44673 < \text{root} < 1.$

And $f(x_1) = -2.17798$ in (1) we get
\[ x_4 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 0.44673 + \frac{1 - 0.44673}{-2.17798 - 0.20356} (0.20356) \quad \ldots (1) \]

\[ \approx 0.44673 + \frac{0.55327}{0.38154} (0.20356) \]

\[ x_4 \approx 0.49402 \]

\[ x_4 = 0.49402 \quad x_5 = 0.50995 \]

\[ X_6 = 0.51520 \quad x_7 = 0.51692 \]

\[ x_8 = 0.52748 \quad x_9 = 0.51767 \]

\[ x_{10} = 0.51775 \quad \text{so on.} \]

\[ \therefore \text{the root is 0.5177 correct to 4 decimals.} \]

Hence the required solution.
METHOD–III : NEWTON – RAPHSON METHOD

The newton – Raphson method is a powerful and elegant method to find the root of an equation.

Let the equation be \( f(x) = 0 \). Let \( x_0 \) be an approximate value of the desired root and \( h \) be the small correction to it so that \( x_1 = x_0 + h \) ........................(1)

Is the root of the equation \( f(x) = 0 \)

\( f(x_1) = 0 \Rightarrow f(x_0 + h) = 0 \)

Expanding \( f(x_0 + h) \) by Taylor’s theorem, we get

\[ f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0 \]

Since \( h \) is small, neglect \( h^2 \) and higher order of \( h \), to get

\[ f(x_0) + h f'(x_0) = 0 \]

\[ h = \frac{-f(x_0)}{f'(x_0)} \] .......................... (2)

On substitution (2) in (1), we get the first approximation to the required root as

\[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \] .......................... (3)

Successive approximations can be written as

\[ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \] .......................... (4) ; \[ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \] .......................... (5)

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \] .......................... (6)

The sequence \( \{x_n\} \) if it converges gives the root. The Newton – Raphson method is also known as Newton’s method of tangent.

Note 1: This method fails if \( f'(x) = 0 \)

Note 2: the initial approximation should be taken very close to the root, otherwise the method may diverge.

Note 3: Newton – Raphson method can be used to find complex root if \( x_0 \) is complex.

Note 4: Newton’s formula converges if \( |f(x)f''(x)| < |f'(x)|^2 \).
Solution of non-linear simultaneous Equations – Newton – Raphson Method:

Consider the equations \( f(x, y) = 0, g(x, y) = 0 \) ........................................ (1)

If an initial approximation \((x_0, y_0)\) to a solution has been found by graphical method or otherwise, then a better approximation \((x_1, y_1)\) can be as below:

Let \( x = x_0 + h, y = y_0 + k \) \( \Rightarrow \) \( f(x_0+h, y_0+k) = 0, \ f(x_0+h, y_0+k) = 0 \)

....................... (2)

On expansion each of the function in (2) by Taylor’s series theorem to first degree terms

\[
\begin{align*}
0 &= f_x(x_0, y_0)x + f_y(x_0, y_0)y + \frac{\partial^2 f}{\partial x^2}(x_0, y_0)x^2 + \frac{\partial^2 f}{\partial y^2}(x_0, y_0)y^2 + \frac{\partial^2 f}{\partial x\partial y}(x_0, y_0)xy \quad \text{etc.}, \\
0 &= g_x(x_0, y_0)x + g_y(x_0, y_0)y + \frac{\partial^2 g}{\partial x^2}(x_0, y_0)x^2 + \frac{\partial^2 g}{\partial y^2}(x_0, y_0)y^2 + \frac{\partial^2 g}{\partial x\partial y}(x_0, y_0)xy \quad \text{etc.},
\end{align*}
\]

Where \( f_0(x_0, y_0) = 0, \ \frac{\partial f}{\partial x}(x_0, y_0) = \left( \frac{\partial f}{\partial x} \right)_{x_0, y_0} \) etc.,

On solving equations (3) for \( h, k \) we get new approximations to the root as

\[ x_1 = x_0 + h, \ y_1 = y_0 + k \]

On continuation process till we get the desired degree of accuracy.
Problem #01 Find the root of \(x^4 - x - 10 = 0\) by using Newton–Raphson Method?
[Ans. \(x_2 = 1.855587\)]

Problem #02 Find the approximation root of \(2x - \log_{10} x - 7 = 0\) by using Newton – Raphson Method?
[Ans. \(x_2 = 3.789278 \approx 3.7893\)]

Problem #03 Solve the system of non-linear equations \(x^2 + y = 11\); \(y^2 + x = 7\)?
[Ans. \(x_2 = 3.5844, y_2 = -1.8482\)]

Problem #04 By Newton–Raphson method to solve the equations \(x = x^2 + y^2\), \(y = x^2 - y^2\) correct to two decimals starting with approximation \((0.8, 0.4)\)?
[Ans. \(x = 0.7974, y = 0.4006\)]

Problem #05 By Newton–Raphson method to solve non-linear system of the equations \(x^2 - y^2 = 4\), \(x^2 + y^2 = 16\) with \(x_0 = y_0 = 2.828\) using Newton Raphson Method. Carry out to two iterations?
[Ans. \(x = 3.162, y = 6.450\)]

Problem #06 By Newton–Raphson method to solve non-linear system of the equations \(x = 2(y + 1)\); \(y^2 = 3xy - 7\) using Newton Raphson Method. Correct to three decimals?
[Ans. \(x = -1.853; y = -1.927\)]

Problem #07 Find Approximation root of the following and check with answers – try yourself?

<table>
<thead>
<tr>
<th>SL NO</th>
<th>function</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(e^x = \sin x)</td>
<td>-0.42036</td>
</tr>
<tr>
<td>2</td>
<td>(x^3 - 25 = 0)</td>
<td>0.5885</td>
</tr>
<tr>
<td>3</td>
<td>(x^4 - x - 1 = 0)</td>
<td>2.924</td>
</tr>
<tr>
<td>4</td>
<td>(x + e^x = 0)</td>
<td>1.22138</td>
</tr>
<tr>
<td>5</td>
<td>(3x^3 + 5x - 40 = 0)</td>
<td>-0.567</td>
</tr>
<tr>
<td>6</td>
<td>(x - e^{-x} = 0)</td>
<td>0.5635</td>
</tr>
<tr>
<td>7</td>
<td>(2 \sin x = x)</td>
<td>0.5671</td>
</tr>
<tr>
<td>8</td>
<td>(x^2 + 4 \sin x = 0)</td>
<td>1.895494</td>
</tr>
<tr>
<td>9</td>
<td>(x^3 - 3x - 5 = 0)</td>
<td>-1.9338</td>
</tr>
<tr>
<td>10</td>
<td>(x^4 - 3x^2 - 5 = 0)</td>
<td>2.7984</td>
</tr>
</tbody>
</table>
Problem #01 Find the root of \( x^4 - x = 10 \) by using Newton – Raphson Method?

Solution: Given \( f(x) \equiv x^4 - x - 10 = 0 \). To find the root of equation by N-R M.

By giving values of \( x \) is 0, 1, 2 till we get \( f(x) \) opposite signs as below:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>1.5</th>
<th>1.75</th>
<th>1.8</th>
<th>1.9</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = x^4 - x - 10 )</td>
<td>-10</td>
<td>-10</td>
<td>-6.4375</td>
<td>-2.3711</td>
<td>-1.3024</td>
<td>1.1321</td>
<td>+4</td>
</tr>
</tbody>
</table>

\[ \therefore \text{The root lies between 1 and 2 i.e., } 1 < \text{root} < 2. \]

\[ \therefore \text{For Further accuracy we observe } f(x) \text{ by giving values as below:} \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>1.5</th>
<th>1.75</th>
<th>1.8</th>
<th>1.9</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-10</td>
<td>-10</td>
<td>-6.4375</td>
<td>-2.3711</td>
<td>-1.3024</td>
<td>1.1321</td>
<td>+4</td>
</tr>
</tbody>
</table>

On observation \( f(x) \) values having opposite signs for the values of \( x \) 1.8 and 1.9.

\[ \therefore \text{The root lies between 1.8 and 1.9 i.e., } 1.8 < \text{root} < 1.9. \]

Now, \( f'(x) = 4x^3 - 1; \therefore x_{n+1} = x_n - \frac{x_n^4 - x_n - 10}{4x_n^3 - 1} \)

Since, \( f \) and \( f'' \) have the same sign at \( x = 1.9 \) choose \( x_0 = 1.9 \) as the starting point.

Now \( x_1 = x_0 - \frac{x_0^4 - x_0 - 10}{4x_0^3 - 1} \)

\[ = 1.9 - \frac{(1.9)^4 - 1.9 - 10}{4(1.9)^3 - 1} \approx 1.9 - \frac{1.1321}{26.436} \approx 1.9 - 0.042824 \approx 1.8572 \]

\( f(x_1) = (1.8572)^4 - (1.8572) - 10 = 11.89692435 - 1.8572 - 10 = 0.03972 \)

\( x_2 = 1.8572 - \frac{0.03972}{24.623} \approx 1.8572 - 0.0016131 \approx 1.85558695 \)

\[ \therefore x_2 = 1.855587 \text{ Find } f(x_2) = (1.85587)^4 - (1.85587) - 10 = +0.000058169 \]

\[ \therefore f(x_2) = +0.000058169 \]

Hence \( x_2 = 1.855587 \) is required desired degree of accuracy solution.
**Problem #02** Find the approximation root of \(2x - \log_{10}x - 7 = 0\) by using Newton – Raphson Method?

[Ans. \(x_2 = 3.789278 \approx 3.7893\)]

**Solution:** Given \(f(x) \equiv 2x - \log_{10}x - 7 = 0\). In \(f(x)\) log function involved and to find the root of equation by N-R M.

By giving values of \(x\) is \(1, 2, 3\) and \(4\) till we get \(f(x)\) opposite signs as below:

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x) = 2x - \log_{10}x - 7)</td>
<td>-5</td>
<td>-3.301</td>
<td>-1.4771</td>
<td>0.3979</td>
</tr>
</tbody>
</table>

\(\therefore\) The root lies between 3 and 4 i.e., \(3 < \text{root} < 4\).

\(\therefore\) For Further accuracy we observe \(f(x)\) by giving \(x\) values as below:

<table>
<thead>
<tr>
<th>(x)</th>
<th>3.5</th>
<th>3.7</th>
<th>3.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x) = 2x - \log_{10}x - 7)</td>
<td>-0.5441</td>
<td>-0.1682</td>
<td>0.0202</td>
</tr>
</tbody>
</table>

On observation \(f(x)\) values having opposite signs for the values of \(x\) 3.7 and 3.8.

\(\therefore\) The root lies between 3.7 and 3.8 i.e., \(3.7 < \text{root} < 3.8\).

Now, \(f'(x) = 2 - \frac{\log_{10}e}{x}; f''(x) = \frac{\log_{10}e}{x^2} = \frac{0.4343}{x^2}\);

Since, \(f\) and \(f''\) have the same sign at \(x = 3.8\) choose \(x_0 = 3.8\) as the starting point. Then \(f(x_0) = 0.0202\).

\(\therefore\) \(x_{n+1} = x_n - \frac{f(x)}{f'(x)} = x_n - \frac{2x_n - \log_{10}e^n - 7}{2 - \frac{\log_{10}e^n}{x_n}}\)

Taking \(n = 0, x_1 = 3.8 - \frac{0.0202}{1.88571} \approx 3.8 - 0.010712 \approx 3.7893\)

\(\therefore x_1 = 3.7893\)

\(f(x_1) = 2(3.7893) - \log_{10}(3.7893) - 7 = 0.000041\)

\(x_2 = 3.7893 - \frac{0.000041}{1.88538} \approx 3.7893 - 0.00002175 \approx 3.789278\)

\(\therefore x_2 = 3.7893\)

Hence \(x_2 = 3.7893\) is required desired degree of accuracy solution.
**Problem #03** Solve the system of non-linear equations \( x^2 + y = 11; y^2 + x = 7 \)?

[Ans. \( x_2 = 3.5844, y_2 = -1.8482 \)]

**Solution:** An initial approximation to the solution is obtained from a rough graph of the given equations as \( x_0 = 3.5 \) and \( y_0 = -1.8 \).

We have \( f(x, y) = x^2 + y - 11 \Rightarrow \frac{\partial f}{\partial x} = 2x; \frac{\partial f}{\partial y} = 1 \)

\( ; \ g(x, y) y^2 + x - 7 \Rightarrow \frac{\partial g}{\partial x} = 1; \frac{\partial g}{\partial y} = 2y \)

The Newton – Raphson non-linear equations

\[
\begin{align*}
    f_0 + h \frac{\partial f}{\partial x_0} + k \frac{\partial f}{\partial y_0} &= 0; \\
    g_0 + h \frac{\partial g}{\partial x_0} + k \frac{\partial g}{\partial y_0} &= 0; \text{ where } f_0 = f(x_0, y_0) \text{ and } \frac{\partial f}{\partial x_0} = \left( \frac{\partial f}{\partial x} \right)_{x_0,y_0}
\end{align*}
\]

Will be \( 7h + k = 0.55; h - 3.6 k = 0.26 \)

On solving, \( h = 0.0855; k = -0.0485 \)

\( \therefore \) the better approximation to the root is \( x_1 = x_0 + h = 3.5855 \)
\( y_1 = y_0 + k = -0.0485 \)

On repeating this process, replacing \( (x_0, y_0) \) by \( (x_1, y_1) \) we get \( (x_2, y_2) \) as \( x_2 = 3.5844 \) and \( y_2 = -1.8482 \)

**Hence** \( x_2 = 3.5844, y_2 = -1.8482 \) is desired degree of accuracy solution.